THE CHOW RING OF RELATIVE FULTON–MACPHERSON SPACE

FUMITOSHI SATO

Abstract. Suppose that X is a nonsingular variety and D is a nonsingular proper subvariety. Configuration spaces of distinct and non-distinct n points in X away from D were constructed by the author and B. Kim in [4] by using the method of wonderful compactification. In this paper, we give an explicit presentation of Chow motives and Chow rings of these configuration spaces.

1. Introduction

Let X be a complex connected nonsingular algebraic variety and let D be a smooth divisor.

In [4], two generalizations of Fulton–MacPherson spaces were constructed by using the method of wonderful compactifications [5]. Two spaces are following:

(1) A compactification $X_D^{[n]}$ of the configuration space of n labeled points in $X \setminus D$, i.e. "not allowing those points to meets D."
(2) A compactification $X_D[n]$ of the configuration spaces of n distinct labeled points in $X \setminus D$, i.e. "not allowing those points to meet each other as well as D."

The goal of this paper is to give an explicit presentation of Chow motives and Chow rings of these configuration spaces. Our main theorems are:

Theorem 1.1. *The Chow ring $A^*(X_D^{[n]})$ is isomorphic to the polynomial ring $A^*(X^n)[x_S]$ modulo the ideal generated by*

(1) $x_S \cdot x_T$ *for* S, T *that overlap,*
(2) $\mathcal{J}_{D_S/X^n} \cdot x_S$ *for all* S,
(3) $P_{D_S/X^n}(-\Sigma_{S' \supset S} x_{S'})$ *for all* S.

Theorem 1.2. *The Chow ring $A^*(X_D[n])$ is isomorphic to the polynomial ring $A^*(X^n)[x_S, y_I]$ modulo the ideal generated by*

(1) $y_I \cdot y_J$ *for* I *and* J *that overlap,*
(2) $x_S \cdot x_T$ *for* S *and* T *that overlap,*
(3) $x_S \cdot y_I$ *unless* $I \subset S$,

(4) $\mathcal{J}_{\Delta_I/X^n} \cdot y_I$ *for all* I,
(5) $\mathcal{J}_{D_S/X^n} \cdot x_S$ *for all* S,
(6) $c_{a,b}(\sum_{a,b\in I} y_I)$ *for* $a, b \in \{1, \cdots, n\}$ *(distinct),*
(7) $P_{D_S/X^n}(-\Sigma_{S'\supset S} x_{S'})$ *for all* S.

The paper is organized as follows. In section 2, we review theory of wonderful compactification and Chow rings and motives after blow-up. In section 3, we review the construction of compactifications of n points in $X\setminus D$. In section 4, we compute Chow groups and motives explicitly. In section 5, we compute Chow rings under the assumptions such that X^n has the Kunneth decomposition and the embedding $D \hookrightarrow X$ is a Lefshetz embedding.

1.1. **Notation.**

- As in [1], for a subset I of $N := \{1, 2, ..., n\}$, let
$$I^+ := I \cup \{n+1\}.$$
- Let Y_1 be the blowup of a nonsingular complex variety Y_0 along a nonsingular closed subvariety Z. If V is an irreducible subvariety of Y_0, we will use $\widetilde{V}$ or $V(Y_1)$ to denote
  - the total transform of V, if $V \subset Z$;
  - the proper transform of V, otherwise.

  If there is no risk to cause confusion, we will use simply V to denote $\widetilde{V}$. The space $\mathrm{Bl}_{\widetilde{V}} Y_1$ will be called the iterated blowup of Y_0 along centers Z, V (with the order).
- For a partition of I of N, Δ_I denotes the polydiagonal associated to I. And consider the binary operation $I \wedge J$ on the set of all partitions satisfying
$$\Delta_I \cap \Delta_J = \Delta_{I\wedge J}.$$
  We use Δ_{I_0} instead of Δ_I when $I = \{I_0, I_1, ..., I_l\}$ such that $|I_i| = 1$ for all $i \geq 1$.

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2. Wonderful Compactification of Arrangements of Subvarieties

In this section, we review the theory of wonderful compactification of arrangements of subvarieties. See the detail and proofs in [5], [6].

2.1. **Arrangement, building set and nest.**

Definition 2.1 (of clean intersection)**.** *Let Y be a nonsingular algebraic variety and let U and V be two smooth subvarieties of Y.*

U and V intersect cleanly if $U \neq V$ and their scheme-theoretic intersection is nonsingular and the tangent bundles satisfy $T(U \cap V) = TU \cap TV$.

Remark 2.2. If the intersection is transversal, then it is a clean intersection.

Definition 2.3 (of arrangement)**.** *A simple arrangement of subvarieties of Y is a finite set $\mathcal{S} = \{S_i\}$ of nonsingular closed irreducible subvarieties of Y satisfying the following conditions*

(1) *S_i and S_j intersect cleanly,*
(2) *$S_i \cap S_j$ is either empty or some S_k's.*

Definition 2.4 (of building set)**.** *Let $\mathcal{S}$ be an arrangement of subvarieties of Y. A subset $\mathcal{G} \subset \mathcal{S}$ is called a building set with respect to $\mathcal{S}$, if , for any $S \in \mathcal{S}$, the minimal elements in $\mathcal{G}$ which contain S intersect transversally and their intersection is S. These minimal elements are called the $\mathcal{G}$-factors of S.*

Definition 2.5 (of $\mathcal{G}$-nest)**.** *A subset $\mathcal{T} \subset \mathcal{G}$ is called a $\mathcal{G}$-nest if there is a flag of elements in $\mathcal{S}$; $S_1 \subset S_2 \subset \cdots \subset S_k$ such that*

$$\mathcal{T} = \cup_{i=1}^{k} \{A : A \text{ is a } \mathcal{G}\text{-factor of } S_i\ \}.$$

2.2. **Construction of $Y_{\mathcal{G}}$ by a sequence of blow-ups.**

Let Y be a nonsingular algebraic variety, $\mathcal{S}$ be a simple arrangement of subvarieties and $\mathcal{G}$ be a building set with respect to $\mathcal{S}$. Order $\mathcal{G} = \{G_1, \cdots, G_N\}$ such that $i < j$ if $G_i \subset G_j$.

We define $(Y_k, \mathcal{S}^{(k)}, \mathcal{G}^{(k)})$ inductively, where Y_k is a blow-up of Y_{k-1} along a nonsingular variety, $\mathcal{S}^{(k)}$ is a simple arrangement of subvarieties of Y_k and $\mathcal{G}^{(k)}$ is a building set with respect to $\mathcal{S}^{(k)}$.

Definition/Theorem 2.6. *Assume $\mathcal{S}$ is a simple arrangement of subvarieties of Y and $\mathcal{G}$ is a building set. Let G be a minimal element in $\mathcal{G}$ and consider $\pi : \widetilde{Y} := \mathrm{Bl}_G Y \to Y$. Denote the exceptional divisor by E. For any nonsingular variety V in Y, we define $\widetilde{V} \subset \mathrm{Bl}_G Y$, the $\sim$ transform of V, to be the proper transform of V if $V \nsubseteq G$, and to be $\pi^{-1}(V)$ if $V \subset G$.*

For simplicity of notation, for a sequence of blow-ups, we use the same notation $\widetilde{V}$ to denote the iterated one.

(1) *The collection $\mathcal{S}'$ of subvarieties in $\widetilde{Y}$ defined by*

$$\mathcal{S}' := \{\widetilde{S}\}_{S\in\mathcal{S}} \cup \{\widetilde{S}\cap E\}_{\emptyset\subsetneq S\cap G\subsetneq S}$$

is a simple arrangement in $\tilde{Y}$

(2) $\mathcal{G}' := \{\widetilde{G_i}\}_{G_i\in\mathcal{G}}$ *is a building set with respect to $\mathcal{S}'$.*

(3) *Given a subset $\mathcal{T}$ of $\mathcal{G}$. Define $\mathcal{T}' := \{\widetilde{A}\}_{A\in\mathcal{T}} \subset \mathcal{G}'$. $\mathcal{T}$ is a $\mathcal{G}$-nest if and only if $\mathcal{T}'$ is a $\mathcal{G}'$-nest.*

Let's go back to the construction of $Y_{\mathcal{G}}$.

(1) For $k = 0$, $Y_0 = Y, \mathcal{S}^{(0)} = \mathcal{S}, \mathcal{G}^{(0)} = \mathcal{G} = \{G_1, \cdots, G_N\}, G_i^{(0)} = G_i$.

(2) Assume Y_{k-1} is already constructed. Let Y_k be the blow-up of $\underbrace{Y_{k-1}}$ along the nonsingular subvariety $G_k^{(k-1)}$. Define $G_i^{(k)} := \widetilde{G_i^{(k-1)}}$. Since $G_i^{(k-1)}$ for $i < k$ are all divisors, $G_k^{(k-1)}$ is minimal in $\mathcal{G}^{(k-1)}$. Thus there is a naturally induced arrangement $\mathcal{S}^{(k)}$ and a building set $\mathcal{G}^{(k)}$ by the theorem 2.6.

(3) Continue the inductive construction to $k = N$, where all elements in the building set $\mathcal{G}^{(N)}$ are divisors.

Theorem 2.7. *Denote $Y^\circ = Y \setminus \cup_{G\in\mathcal{G}}G$. There is a natural locally closed embedding*

$$Y^\circ \hookrightarrow Y \times \prod_{G\in\mathcal{G}} \mathrm{Bl}_G Y,$$

and its closure is denoted by $Y_{\mathcal{G}}$ and called the wonderful compactification of $\mathcal{G}$. Then $Y_{\mathcal{G}}$ is isomorphic to Y_N which is constructed in the above.

The variety $Y_{\mathcal{G}}$ is nonsingular. For each $G \in \mathcal{G}$, there is a nonsingular divisors $D_G \subset Y_{\mathcal{G}}$ such that

(1) *The union of these divisors is $Y_{\mathcal{G}} \setminus Y^\circ$.*

(2) *Any set of these divisors meets transversally. An intersection of divisors $D_{T_1} \cap \cdots D_{T_l}$ is not empty exactly when $\{T_1, \cdots T_l\}$ form a $\mathcal{G}$-nest.*

Theorem 2.8 (order of blow-ups)**.** (1) *Let $\mathcal{I}_i$ be the ideal sheaf of $G_i \in \mathcal{G}$. Then*

$$Y_{\mathcal{G}} \cong \mathrm{Bl}_{\mathcal{I}_1\cdots\mathcal{I}_N} Y.$$

(2) *If we arrange $\mathcal{G} = \{G_1, \cdots G_N\}$ in such an order that*

($*$) *for any $1 \le i \le N$, the first i terms $G_1, \cdots G_i$ form a building set*

Then

$$Y_{\mathcal{G}} \cong \mathrm{Bl}_{\widetilde{G_N}} \cdots \mathrm{Bl}_{\widetilde{G_2}} \mathrm{Bl}_{G_1} Y,$$

where each blow-up is along a smooth subvariety.

2.3. **Chow group and motive of $Y_{\mathcal{G}}$.** Let $Y_0 := Y, Y_0\mathcal{T} := \cap_{T\in\mathcal{T}} T$ where $\mathcal{T}$ is a $\mathcal{G}$-nest. Define $r_{\mathcal{T}}(G) := \dim(\cap_{G\subsetneq T\in\mathcal{T}} T) - \dim G$ (here we use a convention that $\cap_{G\subsetneq T\in\mathcal{T}} T = Y$ if no T strictly contains G). Then define

$$M_{\mathcal{T}} := \{\overrightarrow{\mu} = \{\mu_G\}_{G\in\mathcal{T}} : 1 \leq \mu_G \leq r_{\mathcal{T}}(G) - 1\}$$

and let $\|\overrightarrow{\mu}\| := \sum_{G\in\mathcal{G}} \mu_G$ for $\overrightarrow{\mu} \in M_{\mathcal{T}}$.

Theorem 2.9. *We have the Chow group decomposition*

$$A^*(Y_{\mathcal{G}}) = A^*(Y) \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\overrightarrow{\mu}\in M_{\mathcal{T}}} A^{*-\|\overrightarrow{\mu}\|}(Y_0\mathcal{T})$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

If Y is complete, we also have the Chow motive decomposition

$$h(Y_{\mathcal{G}}) = h(Y) \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\overrightarrow{\mu}\in M_{\mathcal{T}}} h(Y_0\mathcal{T})(\|\overrightarrow{\mu}\|)$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

2.4. **Chow ring of $Y_{\mathcal{S}}$.** In this section, we will review the result of Chow rings after blow-up [3] and the result of Hu [2] concerning the Chow ring of $Y_{\mathcal{S}}$.

Definition 2.10 (of Lefschetz embedding). *An embedding $U \hookrightarrow Y$ is called a Lefshetz embedding if the restriction map $A^*(Y) \to A^*(U)$ is surjective. Under this situation, let $\mathcal{J}_{U/Y}$ be the kernel of $A^*(Y) \to A^*(U)$ and let $P_{U/Y}$ be the Chern polynomial for the normal bundle $N_{U/Y}$.*

Definition 2.11. *A Chern polynomial $P_{U/Y}(t)$ for a Lefshetz embedding $U \hookrightarrow Y$ is a polynomial*

$$P_{U/Y}(t) = t^d + a_1 t^{d-1} + \cdots a_{d-1}t + a_d \in A^*(Y)[t],$$

where d is the codimension of U in Y and $a_i \in A^i(Y)$ is a class whose restriction in $A^i(U)$ is the Chern class $c_i(N_{U/Y})$.

Lemma 2.12. (1) *If D is a divisor, then $P_{D/Y}(t) = t + D$,*
(2) *If $V_1, V_2 \subset Y$ are subvarieties meeting transversally and their intersection is Z, then*

$$P_{Z/Y}(t) = P_{V_1/Y}(t) \cdot P_{V_2/Y}(t)$$

Lemma 2.13. *Let U and V are non-singular closed subvarieties of Y meeting cleanly in a non-singular closed subvariety Z. We also assume that both embeddings $U \hookrightarrow Y$ and $V \hookrightarrow Y$ are Lefschetz. Then all the relevant inclusions below are Lefschetz and*

(1) $\mathcal{J}_{\mathrm{Bl}_Z U/\mathrm{Bl}_V Y} = \mathcal{J}_{U/Y}$ *if* $Z \neq \emptyset$,
(2) $\mathcal{J}_{\mathrm{Bl}_Z U/\mathrm{Bl}_V Y} = (\mathcal{J}_{U/Y}, \widetilde{V})$ *if* $Z = \emptyset$, *where* $\widetilde{V}$ *is the exceptional divisor in* $\mathrm{Bl}_V Y$,
(3) $\mathcal{J}_{\mathrm{Bl}_Z U/\mathrm{Bl}_Z Y} = (\mathcal{J}_{U/Y}, [\mathrm{Bl}_Z V])$ *if* $Z \neq \emptyset$,
(4) $P_{\mathrm{Bl}_Z U/\mathrm{Bl}_V Y}(t) = P_{U/Y}(t)$,
(5) $P_{\mathrm{Bl}_Z U/\mathrm{Bl}_Z Y}(t) = P_{U/Y}(t - \widetilde{Z})$ *where* $\widetilde{Z}$ *is the exceptional divisor in* $\mathrm{Bl}_Z Y$.

Lemma 2.14. *Let* $\{U_i\}$ *be disjoint non-singular closed subvarieties of a smooth variety* Y, *such that* $U_i \hookrightarrow Y$ *are Lefschetz. Then the Chow ring* $A^*(\mathrm{Bl}_{\cup U_i} Y)$ *is isomorphic the polynomial ring* $A^*(Y)[x_i]$ *,where* x_i *corresponds to the exceptional divisor* $\widetilde{U_i}$, *modulo the ideal generated by*
(1) $x_i \cdot x_j$ *for* $i \neq j$,
(2) $\mathcal{J}_{U_i/Y} \cdot x_i$ *for all* i,
(3) $P_{U_i/Y}(-x_i)$ *for all* i.

Definition 2.15. *A regular simple arrangement* $\mathcal{S}$ *is a simple arrangement such that for any* $S_l \subset S_i$, *there is* $S_j \supset S_l$ *such that* $S_l = S_i \cap S_j$.

Theorem 2.16. *Let* $\mathcal{S}$ *be a regular simple arrangement of subvarieties such that all the inclusions* $S_i \subset S_j$ *and* $S_i \subset Y$ *are Lefschetz embedding. Then the Chow ring of* $Y_{\mathcal{S}}$ *is isomorphic to the polynomial ring* $A^*(Y)[x_{S_1}, \cdots, x_{S_N}]$ *(where* x_{S_i} *corresponds to the exceptional divisor* $S_i^{(i+1)}$*) modulo the ideal generated by*
(1) $x_{S_i} \cdot x_{S_j}$ *for incomparable* S_i, S_j,
(2) $\mathcal{J}_{S_i/Y} \cdot x_{S_i}$ *for all* i,
(3) $P_{S_i/Y}(-\Sigma_{S_j \subseteq S_i} x_{S_j})$ *for all* i.

3. Construction of $X_D^{[n]}$ and $X_D[n]$

Fix a nonsingular divisor D of an algebraic variety X of dimension m. In this section, we review constructions of a compactification of configuration spaces of n point in $X \setminus D$, $X_D^{[n]}$, and a compactification of configuration spaces of n distinct point in $X \setminus D$, $X_D[n]$. In this paper, we assume that D is a divisor but every thing will work in the case of D is a smooth subvariety after some adjustment. See the details in [4].

3.1. **Construction.** For a subset S of $N := \{1, 2, ..., n\}$ define a nonsingular subvariety in X^n

$$D_S := \{\mathbf{x} \in X^n \mid \mathbf{x}_i \in D, \ \forall\ i \in S\}.$$

Let $\mathcal{A}$ be the collection of D_S for all $S \subset N := \{1, ..., n\}$ with $|S| \geq 2$. It is clear that the collection is a simple arrangement of smooth

subvarieties of X^n and take a building set $\mathcal{G} = \mathcal{A}$. Then define $X_D^{[n]}$ to be the closure of $X^n \setminus \bigcup_S D_S$ in

$$X^n \times \prod_S \mathrm{Bl}_{D_S} X^n$$

It can be constructed by a successive blowups by theorem 2.7. In particular we may order $\mathcal{G}$ as $D_{12}, D_{123}; D_{13}, D_{23};...;\ D_{12...,n};\ D_{U\cup\{n\}}$ with $|U| = n-2$ and $U \subset N \setminus \{n\};...;\ D_{in}$ for $i = 1, ..., n-1$ by theorem 2.8.

Lemma 3.1. *Let I_1 and J_2 be partitions of N. The intersection of proper transforms of Δ_{I_1} and Δ_{I_2} is the proper transform of the intersection $\Delta_{I_1 \wedge I_2}$.*

Corollary 3.2. *For $I \subset N$ with $|I| \geq 2$, $\Delta_I(X_D^{[n]})$ form a building set of nonsingular subvarieties of $X_D^{[n]}$ with respect to the set of all polydiagonals.*

Definition 3.3. *Define $X_D[n]$ to be the closure of $X_D^{[n]} \setminus \bigcup_{|I|\geq 2} \Delta_I(X_D^{[n]})$*

$$X_D^{[n]} \times \prod_{|I|\geq 2} \mathrm{Bl}_{\Delta_I(X_D^{[n]})} X_D^{[n]}$$

Theorem 3.4. (1) *$X_D[n]$ is a nonsingular variety. There is a natural projection from $X_D[N]$ to $X_D[I]$ for any subset I of N. There is a natural S_n-action on $X_D[n]$.*

(2) *The boundary is the union of divisors $\widetilde{D_S}$ with $|S| \geq 1$, and $\widetilde{\Delta_I}$ with $|I| \geq 2$ of normal crossings.*

(3) *The intersections of boundary divisors are nonempty if and only if they are nested. Here $\{D_{S_i}, \Delta_{I_j}\}$ is nested if each pair S_i and S_k (T_j and T_l) is either disjoint or one is contained in the other and each pair S_i and T_k is either disjoint or T_k is contained in S_i.*

(4) *We may take order D_S; Δ_I for $n \notin S, I$; and then D_T with $n \in T$, then Δ_J with $n \in J$.*

4. Chow groups and motives

In this section, we will apply theorem 2.9 to $X_D^{[n]}$ and $X_D[n]$. For simplicity, we assume that X is complete.

4.1. **Chow group and motive of $X_D^{[n]}$.** In this case, our $Y = X^n, \mathcal{S} = \mathcal{G} = \{D_S : S \subset N \text{with} |S| \geqslant 2\}$ where $D_S = \{\mathbf{x} \in X^n \mid \mathbf{x}_i \in D,\ \forall\ i \in S\}$. We have $\mathcal{S} = \mathcal{G}$, so a $\mathcal{G}$-nest is just a chain of elements in $\mathcal{S}$, $\mathcal{T} = \{D_{S_1} \subset D_{S_2} \subset \cdots \subset D_{S_k}\}$. Thus $Y_0\mathcal{T} = D_{S_1}$.

A chain $\mathcal{CH}$ is a chain of subset of N, $S_k \subsetneq \cdots \subsetneq S_2 \subsetneq S_1$, such that S_k is not a singleton. Obviously, there is one-to one correspondence between a set of chains of $\mathcal{S}$ and a set of chains of N. We say $\emptyset$ is also a chain. We define $\max_{\mathcal{CH}(\mathcal{T})} S$ as the maximal element of $\mathcal{CH}(\mathcal{T})$ which is strictly contained in S, where $\mathcal{CH}(\mathcal{T})$ is the chain of N which corresponds to $\mathcal{T}$. If there is no such element, then we define $\max_{\mathcal{CH}(\mathcal{T})} S = \emptyset$

Now let $G = D_S$ and let's compute $r_{\mathcal{T}}(G)$;

$$\begin{aligned} r_{\mathcal{T}}(G) &= \dim(\bigcap_{G \subsetneq T \in \mathcal{T}} T) - \dim G \\ &= \dim(D_{\max_{\mathcal{CH}(\mathcal{T})} S}) - \dim D_S \\ &= |S| - |\max_{\mathcal{CH}(\mathcal{T})} S|. \end{aligned}$$

Remark 4.1 (When D is not a divisor). When D is not a divisor, then we also blow up $D_{\{i\}}$. So we will not exclude the case such that S_k is a singleton for $\{S_k \subsetneq \cdots \subsetneq S_2 \subsetneq S_1\}$. $r_{\mathcal{T}}(G)$ will be also changed, it will be multiplied by the codimension of D in X.

For a chain $\mathcal{CH}(\neq \emptyset)$, define

$$M_{\mathcal{CH}} := \{\overrightarrow{\mu} = \{\mu_S\}_{S \in \mathcal{CH}} : 1 \leq \mu_S \leq |S| - |\max_{\mathcal{CH}} S| - 1\}.$$

For $\mathcal{CH} = \emptyset$, define $M_{\mathcal{CH}}$ is consist of one $\overrightarrow{\mu}$ with $\|\overrightarrow{\mu}\| = 0$ and $D_\emptyset = X^n$.

Theorem 4.2. *Let X be a complete nonsingular variety. Then we have the Chow group and motive decompositions*

$$A^*(X_D^{[n]}) = \bigoplus_{\mathcal{CH}} \bigoplus_{\overrightarrow{\mu} \in M_{\mathcal{CH}}} A^{*-\|\overrightarrow{\mu}\|}(D_{S_{\mathcal{CH}}}),$$

$$h(X_D^{[n]}) = \bigoplus_{\mathcal{CH}} \bigoplus_{\overrightarrow{\mu} \in M_{\mathcal{CH}}} h(D_{S_{\mathcal{CH}}})(\|\overrightarrow{\mu}\|),$$

where $\mathcal{CH}$ runs through all the chains of N and $S_{\mathcal{CH}}$ is the maximal element in $\mathcal{CH}$.

4.2. **Chow group and motif of $X_D[n]$.** We use the same notation as [6].

(1) We call two subsets $I, J \subset N$ are overlapped if $I \cap J$ is not a nonempty proper subset of both I and J. For a set $\mathcal{N}$ of subsets of N, we call I is compatible with $\mathcal{N}$, denoted by $I \sim \mathcal{N}$, if I does not overlap any elements of $\mathcal{N}$.

A nest $\mathcal{N}$ is a set of subset of N such that any pair $I \neq J \in \mathcal{N}$ are not overlapped and contains all singletons.

For a given nest $\mathcal{N}$, define $\mathcal{N}^\circ := \mathcal{N} \setminus \{\{1\}, \cdots, \{n\}\}$.

A nest $\mathcal{N}$ naturally corresponds to a tree (which may not be connected) with each node is labeled by an element of $\mathcal{N}$. Let $c(\mathcal{N})$ be the number of connected components of the forest which corresponds to $\mathcal{N}$. Denote by $c_I(\mathcal{N})$ the number of maximal elements of the set $\{J \in \mathcal{N} : J \subsetneq I\}$, which is called the number of sons of the node I.

Let $\overline{\Delta_{\mathcal{N}}} := \cap_{I \in \mathcal{N}} \Delta_I(X_D^{[n]})$ in this section.

(2) For a nest $\mathcal{N}(\neq \{\{1\}, \cdots \{n\}\})$, define

$$M_{\mathcal{N}} := \{\overrightarrow{\mu} = \{\mu_I\}_{I \in \mathcal{N}} : 1 \leq \mu_I \leq m(c_I - 1) - 1\}$$

where $m = \dim X$.

For $\mathcal{N} = \{\{1\}, \cdots \{n\}\}$, define $M_{\mathcal{N}} = \{\overrightarrow{\mu}\}$ with $\|\mu\| = 0$.

As in [6], we have

Proposition 4.3. *We have the Chow group and motive decompositions*

$$A^*(X_D[n]) = \bigoplus_{\mathcal{N}} \bigoplus_{\overrightarrow{\mu} \in M_{\mathcal{N}}} A^{*-\|\overrightarrow{\mu}\|}(\overline{\Delta_{\mathcal{N}}}),$$

$$h(X_D[n]) = \bigoplus_{\mathcal{N}} \bigoplus_{\overrightarrow{\mu} \in M_{\mathcal{N}}} h(\overline{\Delta_{\mathcal{N}}})(\|\overrightarrow{\mu}\|),$$

where $\mathcal{N}$ runs through all the nest of N

Now we need to simplify $A^*(\overline{\Delta}_{\mathcal{N}})$ and $h(\overline{\Delta}_{\mathcal{N}})$.

Lemma 4.4. *D_S and Δ_I intersect cleanly.*

Proof. We only need to prove that $TD_S \cap T\Delta_I \subset T(D_S \cap \Delta_I)$. An arc in Δ_I have a coordinate representative $(\mathbf{x}_i) \in X^n$ such that $\mathbf{x_i} = \mathbf{x_j}$ for $i, j \in I$. For an arc in Δ_I to be an arc in D_S, $\mathbf{x}_i \in D$ for all $i \in S$. Thus the arc should be an arc in $D_S \cap \Delta_I$. ☐

Proposition 4.5. *$\overline{\Delta_I}$ is isomorphic to $X_D^{[|I^c|+1]}$.*

Proof. We need to know which blow ups of D_S have an effect to Δ_I in a specific order of blow ups. We can assume that $I = \{l, \cdots, n\}$ by arranging the order and denote $a = |I^c|$ and $b = |I|$. We will denote Δ_I by $X^a \times \Delta(\cong X^{|I^c|+1})$. Then we have two different kinds of D_S. The first one is that $S \subset I^c$, which we call the first kind, the second one is that $S \not\subseteq I^c$, which we call the second kind. We will change the order of blow ups so that we first blow up along D_S of the first kind, and then along the second kind. More precisely, we order $D_{I^c} \times X^b, D_{1,\cdots,\hat{i},\cdots,l} \times X^b, \cdots, D_{i,j} \times X^b (i, j \in \{1, \cdots, a\})$ and then $D_{I^c} \times D^b, \cdots, D_{S'} \times D_{S''}, \cdots (|S''| > 0$ and $(|S'|, |S''|)$:

non-increasing in lexicographical order). This order satisfies $(*)$-condition in definition/theorem 2.6, so that we can blow up in this order.

In this order of blow ups, notice that $X^a \times \Delta$ and $D_{S'} \times D_{S''}$ for $S'' \subsetneq I$ are separated when we blow up along $D_{S'} \times D^b$. Thus we can forget the process of blow ups by $D_{S'} \times D_{S''}$ where $S'' \subsetneq I$ i.e. we only need to care about $D_{S'} \times D^b$ for the second kind. Under the isomorphism $X^a \times \Delta \cong X^{a+1}$, they are just $D_{S'} \times D$.

$\square$

We can also apply the same technique to polydiagonals term by term. Thus we can go further from proposition 4.3.

Theorem 4.6. *We have the Chow group and motive decompositions*

$$A^*(X_D[n]) = \bigoplus_{\mathcal{N}} \bigoplus_{\overrightarrow{\mu} \in M_{\mathcal{N}}} (\bigoplus_{\mathcal{CH}} \bigoplus_{\overrightarrow{\lambda} \in M_{\mathcal{CH}}} A^{*-\|\overrightarrow{\mu}\|-\|\overrightarrow{\lambda}\|}(D_{S_{\mathcal{CH}}})),$$

$$h(X_D[n]) = \bigoplus_{\mathcal{N}} \bigoplus_{\overrightarrow{\mu} \in M_{\mathcal{N}}} (\bigoplus_{\mathcal{CH}} \bigoplus_{\overrightarrow{\lambda} \in M_{\mathcal{CH}}} h(D_{S_{\mathcal{CH}}})(\|\overrightarrow{\mu}\| + \|\overrightarrow{\lambda}\|)),$$

where $\mathcal{N}$ runs through all the nest of $\mathcal{N}$ and $\mathcal{CH}$ runs through all the chains of $c(\mathcal{N})$.

5. Chow rings

In this section we assume that X has a cellular decomposition and D is a smooth divisor of X such that $D \hookrightarrow X$ is a Lefshetz embedding. The reason we assume these conditions is that we need a Kunneth decomposition and S. Keel's formula for intersection ring of blow-up.

5.1. **Chow ring of $X_D^{[n]}$.**

Note that $D_S \hookrightarrow D_{S'}$ for $S \supset S'$ and $D_S \hookrightarrow X^n$ are Lefshetz embedding.

Obviously, the arrangement $\mathcal{A}$ is regular, so we can apply theorem 2.16.

Theorem 5.1. *The Chow ring $A^*(X_D^{[n]})$ is isomorphic to the polynomial ring $A^*(X^n)[x_S]$ modulo the ideal generated by*

(1) $x_S \cdot x_T$ *for S, T that overlap,*
(2) $\mathcal{J}_{D_S/X^n} \cdot x_S$ *for all S,*
(3) $P_{D_S/X^n}(-\Sigma_{S' \supset S} x_{S'})$ *for all S.*

5.2. **Chow ring of $X_D[n]$.** We will compute the Chow ring of $X_D[n]$ from $X_D^{[n]}$ by a sequence of blow ups along, which is same as [1],

$$\Delta_{\{1,2\}}, \Delta_{\{1,2,3\}}, \Delta_{\{1,3\}}, \Delta_{\{2,3\}}, \cdots, \Delta_{\{1,\cdots,n\}}, \cdots, \Delta_{\{1,n\}}, \cdots, \Delta_{\{n-1,n\}}.$$

Let

$$Y_i^{[i]} \to \cdots \to Y_{k+1}^{[i]} \to Y_k^{[i]} \to \cdots \to Y_0^{[i]}$$

be a part of the above sequence of blow-ups along

$$\Delta_{\{1,\cdots,i+1\}}, \cdots \Delta_{\{1,\cdots,i-k-1,i+1\}}, \cdots \Delta_{\{k,\cdots,i,i+1\}}, \cdots, \Delta_{\{1,i+1\}}, \cdots, \Delta_{\{i,i+1\}}.$$

Note $1 \leq i \leq n-1$.

We will compute Chow rings of $Y_k^{[i]}$'s inductively by using theorem 2.14.

Lemma 5.2. *If I' and J' are subsets of $\{1, \cdots, i, i+1\}$ that overlap, then $\widetilde{\Delta_{I'}}$ and $\widetilde{\Delta_{J'}}$ are disjoint at $Y_k^{[i]}$, except, up to the order of I' and J', in exactly the following cases:*

1. *$I' = I \subset \{1, \cdots, i\}, |I| \leq i-k, J' = J^+$, with $J \subset I$,*
2. *$I' = I^+, J' = J^+$, with $I \cap J = \emptyset, |I \cup J| \leq i-k$*

Proof. We change the order of blow ups in the following way;

$$D_S, \Delta_I; D_{S^+}, \Delta_{I^+}; D_{S^{++}},$$

where $S, I \subset \{1, \cdots, i\}, |I^+| \leq i-k+2$ and $S^{++} \not\subseteq \{1, \cdots, i, i+1\}$. After blowing up along D_S, Δ_I, the space is $X_D[i] \times X^{(n-i)}$. If $I', J' \subset \{1, \cdots, i\}$, then $\widetilde{\Delta_{I'}}$ and $\widetilde{\Delta_{J'}}$ are disjoint by theorem 3.4.

For $I' = I^+, J' = J^+$, $\widetilde{\Delta_{I^+}}$ is a product of the graph of $p_a : \overline{\Delta_I} \to X$ and X^{n-i-1} where $a \in I$ and we use a convention $\Delta_a = X^n$. Same for $\widetilde{\Delta_{J^+}}$. To have non-empty intersection, I and J must be nested by theorem 3.4. But we have an assumption that I^+ and J^+ overlap, so that I and J must be disjoint. $\widetilde{\Delta_{I^+}}$ and $\widetilde{\Delta_{J^+}}$ will be separted after blowing up along $\widetilde{\Delta_{(I\cup J)^+}}$.

Now let's move to the case that $I' = I \subset \{1, \cdots, i\}$ and $J' = J^+$. In this case, $\widetilde{\Delta_I} = \overline{\Delta_I} \times X^{n-i}$. To have non-empty intersectoin, I and J are nested, i.e. $J \subset I$ or $I \subset J$. But the latter case $I' \subset J'$, which contradict to the assumption of overlapping. Thus $J \subset I$. $\widetilde{\Delta_I}$ and $\widetilde{\Delta_{J^+}}$ will be separted after blowing up along $\widetilde{\Delta_{I^+}}$.

Note that D_S and Δ_I are intersecting cleanly and its intersection is a proper subset of Δ_I. □

Lemma 5.3. *For $a \in I \subset \{1, \cdots, i\}$ such that $2 \leq |I| \leq i-k$, then at $Y_k^{[i]}$,*

$$\widetilde{\Delta_{I^+}} = \widetilde{\Delta_I} \cap \widetilde{\Delta_{a^+}}.$$

Proof. Proof is very similar to proposition 3.1.

□

Lemma 5.4. *If $\widetilde{\Delta_{I'}}$ is a divisor in $Y_k^{[i]}$, then the inverse image $\pi^*(\widetilde{\Delta_{I'}})$ in $Y_k^{[i]}$ is the divisor $\widetilde{\Delta_{I'}}$, except cases such that $I' = J \subset \{1, \cdots, i\}$ with $|J| = i-k$ and in that case*

$$\pi^*(\widetilde{\Delta_J}) = \widetilde{\Delta_J} + \widetilde{\Delta_{J^+}}.$$

Proof. For the case described in the statement, by lemma 5.3, the statement is true. For other cases, it is obvious that the disisor $\widetilde{\Delta_{I'}}$ does not contain any blow up center by considering the space $X_D^{[i]} \times X^{(n-i)}$. □

For $a \in N$, let p_a be the corresponding projection from X^n to X, and for $a, b \in N$ (distinct), let $p_{a,b}$ be the projection from X^n to $X^{a,b}$. Let $[\Delta] \in A^m(X^{a,b})$ be the class of the diagonal, where $m = \dim X$. Define a polynomial $c_{a,b}(t) \in A^*(X^n)[t]$ be

$$c_{a,b}(t) = \sum_{i=1}^{m} (-1)^i p_a^*(c_{m-i}) t^i + [\Delta_{\{a,b\}}]$$

where c_{m-i} is the $(m-i)$-th Chern class of X and $[\Delta_{\{a,b\}}] = p_{a,b}^*([\Delta])$.

Let's compute Chern polynomials and Lefshetz kernels of Δ's at the stage of $Y_0^{[1]}$.

Lemma 5.5. (1) *$\mathcal{J}_{\Delta_I(Y_0^{[1]})/Y_0^{[1]}} = (\mathcal{J}_{\Delta_I/X^n}, x_S)$ where $S \not\supseteq I$.*

(2) *$P_{\Delta_I(Y_0^{[1]})/Y_0^{[1]}}(t) = P_{\Delta_I/X^n}(t)$.*

Proof. (1) By the proof of proposition 4.5, we know that $\widetilde{D_S}$ for $S \not\supseteq I$ is disjoint from $\widetilde{\Delta_I}$, and others intersect cleanly and non-trivially. By lemma 2.13, we have the statement.

(2) We know that $\widetilde{\Delta_I}$ is intersecting with $\widetilde{D_S}$ cleanly including the cases disjoint by the proof of proposition 4.5. By lemma 2.13, we know that a Chern polynomial will not be changed.

□

Proposition 5.6. (1) *For $a \in \{1, \cdots, i\}, 0 \leq k \leq i-1$, a Chern polynomial of $\widetilde{\Delta_{a^+}}$ at $Y_k^{[i]}$ is*

$$c_{a,i+1}(-t + \sum_{a,i+1 \in I'} D_k I').$$

(2) *For* $I \subset \{1, \cdots, i\}, 2 \leq |I| \leq i-k$, *a Chern polynomial of* $\widetilde{\Delta_{I^+}}$ *at* $Y_k^{[i]}$ *is*

$$(t + D_k I) \cdot c_{a,i+1}(-t + \sum_{I^+ \subset I'} D_k I')$$

for any $a \in I$.

Here $D_k I$ *is the divisor of* $Y_k^{[i]}$ *corresponding to* Δ_I.

Proof. Exactly same as [1]. □

Proposition 5.7. *Let* $I' = I^+ \subset \{1, \cdots, i, i+1\}$ *such that* $|I'| = i-k+1$. *Then the restriction* $\widetilde{\Delta_{I'}} \to Y_k^{[i]}$ *is Lefschetz embedding, and its Lefschetz kernel is generated by*

(1) $D_k J'$ *for any* $J' \subset \{1, \cdots i, i+1\}$ *that overlaps with* I', *except if* $I \subset J' \subset \{1, \cdots i, i+1\}$.
(2) $\mathcal{J}_{\Delta_{I'}/X^n}$.
(3) x_S *for* $S \not\supseteq I'$.

Proof. By lemma 2.13, $\widetilde{\Delta_{I'}} \to Y_k^{[i]}$ is Lefschetz embedding.

Now let's prove the statement for generators. By lemma 2.13, we have to show that, for J' which overlap with I', those exceptional cases are exactly blow up centers which intersect $\widetilde{\Delta_{I'}}$ with non-empty intersection. The order of blow ups does not matter to the statement, so that we can change the order as we want.

First consider a case that $I' \cap J' \neq \emptyset$. We can assume $i+1 \in I' \cap J'$ by changing numbering. In this case, by lemma 5.2, we know exactly when the intersection is non-empty or not.

Now, consider a case that $I' \cap J' = \emptyset$. We can assume that $J' = \{1, \cdots, j\}$ and $I' \subset \{j+1, \cdots, j+i+1\}$. Then by the inductive construction of $X_D[n]$, it is obvious they intersect.

□

Proposition 5.8. *For* $0 \leq k \leq i$, $A^*(Y_k^{[i]})$ *is the polynomial ring* $A^*(X^n)[x_S, D_k I]$, *where* $S \subset N$ *such that* $|S| > 1$ *and* $I \subset \{1, \cdots, i+1\}$ *such that either* $I \subset \{1, \cdots, i\}$ *or* $|I| > i-k+1$, *modulo the ideal generated by*

(1) $D_k I \cdot D_k J$ *for* I *and* J *that overlap,*
(2) $x_S \cdot D_k I$ *unless* $I \subset S$,
(3) $\mathcal{J}_{\Delta_I/X^n} \cdot D_k I$ *for all* I,
(4) (a) $c_{a,b}(\sum_{a,b \in I} D_k I)$ *for* $a, b \in \{1, \cdots, i\}$ *(distinct);*
 (b) $D_k I \cdot c_{a,i+1}(\sum_{I^+ \subset I'} D_k I')$ *for* $I \subset \{1, \cdots, i\}, |I| > i-k, a \in I$ *and* $I^+ = I \cup \{i+1\}$
(5) $x_S \cdot x_T$ *for* S *and* T *that overlap,*

(6) $\mathcal{J}_{D_S/X^n} \cdot x_S$ *for all* S,
(7) $P_{D_S/X^n}(-\Sigma_{S'\supset S} x_{S'})$ *for all* S.

Proof. For $Y_0^{[1]}$, it is just theorem 5.1. Also note that $Y_i^{[i]} = Y_0^{[i+1]}$ and the statement for $Y_i^{[i]}$ will imply $Y_0^{[i+1]}$ because condition (4b) is vacuous when $k = 0$.

We only need to prove that the statement for $Y_k^{[i]}$ will imply the one for$Y_{k+1}^{[i]}$. The conditions (5) to (7) are coming from blow up along D_S and these are not new.

For (4), proof is exactly same as [1].

(1), (2), and (3) follow from proposition 5.7. □

Especially, we have

Theorem 5.9. *The Chow ring* $A^*(X_D[n])$ *is isomorphic to the polynomial ring* $A^*(X^n)[x_S, y_I]$ *modulo the ideal generated by*

(1) $y_I \cdot y_J$ *for* I *and* J *that overlap,*
(2) $x_S \cdot x_T$ *for* S *and* T *that overlap,*
(3) $x_S \cdot y_I$ *unless* $I \subset S$,
(4) $\mathcal{J}_{\Delta_I/X^n} \cdot y_I$ *for all* I,
(5) $\mathcal{J}_{D_S/X^n} \cdot x_S$ *for all* S,
(6) $c_{a,b}(\sum_{a,b\in I} y_I)$ *for* $a, b \in \{1, \cdots, n\}$ *(distinct),*
(7) $P_{D_S/X^n}(-\Sigma_{S'\supset S} x_{S'})$ *for all* S.

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School of Mathematics, Korean Institute for Advanced Study
E-mail address: `fumi@kias.re.kr`